MATCHING EXTENSION AND CONNECTIVITY IN GRAPHS

by

M.D. Plummer*
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37235

dedicated to Frank Harary
on the occasion of his sixty-fifth birthday

1. Introduction and Terminology

All graphs in this paper will be finite and connected and will have no loops or parallel lines.
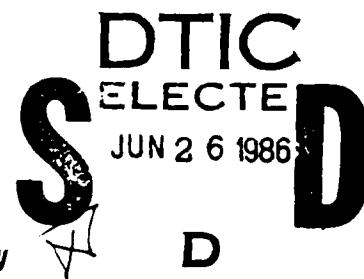
Let n and p be positive integers with $n \leq (p-2)/2$ and let G be a graph with p points having a perfect matching. Graph G is said to be **n-extendable** if every matching of size n in G extends to a perfect matching. In this paper, we will be concerned primarily with studying the relationship between n-extendability and connectivity in graphs.

Let us begin, however, with a few historical remarks. The concept of n-extendability seems to have its earliest roots in a paper of Hetyei (1964) who studied the concept for *bipartite* graphs. In this early paper, Hetyei obtained three different characterizations of 1-extendable bipartite graphs. Lovász and the present author (1977) gave a fourth such characterization which they referred to as an "ear structure theorem". Unknown to them, however, Hartfiel (1970) had already formulated an equivalent theorem, but couched in terms of matrices. A year later, Brualdi and Perfect (1971) published a paper in which they gave the first characterization of n-extendable bipartite graphs, but they too couched their results in terms of matrices ("extending partial diagonals") and set systems ("extending partial systems of distinct representatives (PSDR's)"). For more on bipartite n-extendable graphs, see Plummer (1986a).

The more general family of n-extendable graphs which are not necessarily bipartite seems to have even earlier roots. In the late 1950's, Kotzig (1959a, 1959b, 1960) began to develop a decomposition theory for graphs with perfect matchings, but unfortunately these papers did not receive the attention that they deserve, due to the fact that they were written

in Slovak. In the early 1960's, the study of decompositions of graphs in terms of their maximum matchings was begun by Gallai (1963, 1964) and independently by Edmonds (1965). One of the degenerate cases of their theory for *maximum* matchings, however, arises when the graphs in question have *perfect* matchings.

Motivated by the results of Kotzig, Gallai and Edmonds, Lovász (1972) extended and refined the canonical decompositions already extant while analyzing further the structure of graphs which are *elementary*, thus extending the earlier work of Hetyei and Kotzig. A graph G is called **elementary** if the set of its lines which lie in at least one perfect matching form a connected subgraph of G.

In this same paper, Lovász introduced the concept of a *bicritical* graph. A graph G is said to be **bicritical** if $G - u - v$ has a perfect matching for every pair of distinct points u and v in $V(G)$. In the last ten years or so, the earlier work on decompositions of graphs in terms of their matchings has evolved further (see Lovász and Plummer (1986)) and today much attention continues to be focused upon the structure of bicritical graphs which are, in addition, 3-connected. Such graphs have been christened **bricks**. (See, for example, the paper by Edmonds, Lovász and Pulleyblank (1982).)

But what is the connection between n-extendability and bicriticality? In 1980, the author published a paper on general n-extendable graphs. One of the results presented in that paper states that every 2-extendable graph is either bipartite or is a brick. (The reader should convince himself immediately that these two classes of graphs are disjoint.) Motivated by this result, the author has continued to study properties of n-extendable graphs (see (1985, 1986a, 1986b and 1986c)).

All graph terminology not defined in this paper may be found in Bondy and Murty (1976) and Lovász and Plummer (1986).

2. Connectivity and n-extendability of a Graph

In addition to the theorem of the author found in (1980) and mentioned in the Introduction, there are two other results proved in that paper which we shall need repeatedly and hence we state them without proof.

1980A. THEOREM. *If $n \geq 2$ and G is n-extendable, then G is also $(n - 1)$-extendable.*

1980B. THEOREM. *If G is n-extendable, then G is $(n + 1)$-con-nected.* ∎

However, one can say more about the minimum point cutsets of an n-extendable graph. But first we need the following lemma which is immediate using Philip Hall's classical theorem on bipartite graph matching.

2.1. LEMMA. *Let G be k-connected, let S be a minimum cutset in G, and let C be any component of $G - S$. Then given any subset $S' \subseteq S$, $S' \neq \emptyset$ and $|S'| \leq |V(C)|$, there exists a complete matching of S' into $V(C)$.*

PROOF. Suppose that the conclusion of the Lemma is false. So there exists a non-empty subset S' of S with $|S'| \leq |V(C)|$, but no complete matching of S' into $V(C)$. Consider the bipartite graph B with point set bipartition $V(B) = S' \cup V(C)$ and with $E(B)$ consisting of all lines of G joining S' to $V(C)$.

Applying P. Hall's theorem to the bipartite graph B, there must be a set $S'' \subseteq S'$, with $|\Gamma(S'')| < |S''|$ and $S'' \neq \emptyset$.

Then since $|\Gamma(S'')| < |S''| \leq |S'| \leq |V(C)|$, there must be a point $u \in V(C) - \Gamma(S'')$. But then $T = (S - S'') \cup (\Gamma(S''))$ separates u from any other component C' of $G - S$ where $C' \neq C$. But then $|T| = |S - S''| + |\Gamma(S'')| = |S| - |S''| + |\Gamma(S'')| < |S|$, contradicting the fact that S is a minimum cutset in G. ∎

We are now prepared for the main result of this section.

2.2. THEOREM. *Let G be an n-extendable graph with $n \geq 1$ and let $S \subseteq V(G)$ be a cutset of G with $|S| = n + 1$. Then:*

(a) S is independent.

(b) If in addition, $n \geq 2$, then $G - S$ has at most $n + 1$ components and equality holds if and only if $G = K_{n+1,n+1}$.

PROOF. First note that since $|V(G)| = p \geq 2n + 2 > n + 2$, it follows that $G \neq K_{n+2}$ and hence S is a *non-trivial* cutset; that is, $G - S$ has at least two components. By Theorem 1980B, G is $(n + 1)$-connected and hence S must be a minimum cutset of G.

Suppose S is not independent. Then we may assume that $S = \{a, b, u_1, \ldots, u_{n-1}\}$ where $ab \in E(G)$. Let the components of $G - S$ be $C_1, \ldots, C_r$ where $r > 1$. Relabeling the components if necessary, we may assume that $|V(C_1)| \geq |V(C_j)|$, for $j > 1$.

Claim. For each component C_i of $G - S$, $|V(C_i)| \leq n - 2$.

Suppose, to the contrary, that, say, $|V(C_1)| \geq n - 1$. By the preceding lemma, we can match all of $u_1, \ldots, u_{n-1}$ into $V(C_1)$. Let this matching be $M_1 = \{u_1 v_1, \ldots, u_{n-1} v_{n-1}\}$.

Now $M_1 + ab$ is a matching of size n which must, therefore, extend to a perfect matching F_1 of G. Thus in particular, $|V(C_1)| - (n-1)$ must be a non-negative even integer.

Now u_1 must be adjacent to some point $w_1 \in V(C_2)$. Then $M_2 = M_1 - u_1 v_1 + u_1 w_1 + ab$ must be a matching of size n which cannot be extended to a perfect matching for G since M_2 covers S, but leaves an odd number of points in $V(C_1)$ unmatched. So we have a contradiction of the assumption that G was n-extendable and the Claim is proved.

Now let us suppose that at least one component, say C_1, of $G - S$ contains at least 2 points. Since then by the above Claim, $2 \leq |V(C_1)| \leq n-2 < n-1$, by the preceding Lemma we can match some $|V(C_1)|$ points from $S - a - b$ into (and therefore *onto*) $V(C_1)$. Denote this matching by $M_3 = \{e_1, \ldots, e_t\}$ where $e_i = u_i v_i$ and $t = |V(C_1)|$. Note that M_3 leaves $m = (n-1) - t > 0$ points of $S - a - b$ unmatched.

Suppose now that $m < \sum_{i=2}^{r} |V(C_i)|$. Then by Lemma 2.1, the m points in $S - a - b$ unmatched by M_3 can be matched into $V(C_2) \cup \cdots \cup V(C_r)$. Let $M_4 = \{e_1, \ldots, e_t, e_{t+1}, \ldots, e_{n-1}\}$ be this extension of M_3, where $e_i = u_i v_i$, where $u_i \in S - a - b$.

Suppose one of $u_1, \ldots, u_t$, say u_1, is adjacent to $y \in \bigcup_{i=2}^{r} V(C_i)$ such that y is not covered by matching M_4. Then $M_5 = M_4 - u_1 v_1 + u_1 y + ab$ is a matching of size n which does not extend to a perfect matching of G, since v_1 cannot be matched. This is a contradiction of n-extendability.

So we may suppose that $u_1, \ldots, u_t$ are adjacent only to points in $\bigcup_{i=2}^{r} V(C_i)$ which are covered by matching M_4.

Now we may assume that our matching M_4 is "greedy" in the sense that no point in C_{j+1} is covered by M_4 until *all* points of C_j are covered by M_4. In this way, we see that at most one of the components $C_1, \ldots, C_r$ is partially – but not completely – matched by M_4. (See Figure 1.)

Now suppose M_4 covers all points of $\bigcup_{i=1}^{r} V(C_i)$. Then $|V(G)| = 2 + 2(n-1) = 2n$, a contradiction since G n-extendable implies $p \geq 2n+2$. So there are points in $\bigcup_{i=2}^{r} V(C_i)$ which are not matched by M_4.

Let C_k be a component containing a point v_0 not matched by M_4. Let $|V(C_k)| = s$. Now $t = |V(C_1)| \geq |V(C_k)| = s$, so by the preceding Lemma, there exists a complete matching N of $\{u_1, \ldots, u_s\}$ into, and therefore onto, $V(C_k)$. Hence N must match one of $u_1, \ldots, u_s$ to v_0, say $u_j v_0 \in N$. Hence $M_5 = M_4 + ab - u_j v_j + u_j v_0$ is a matching of size n

FIGURE 1.

which covers all of S and all of C_1, except point v_j. Thus M_5 does not extend to a perfect matching for G, a contradiction.

So we may assume that $m = (n-1) - t \geq \sum_{i=2}^{r} |V(C_i)|$. But then

$$|V(G)| = |S| + \sum_{i=1}^{r} |V(C_i)|$$

$$= (n+1) + |V(C_1)| + \sum_{i=2}^{r} |V(C_i)|$$

$$= n + 1 + t + \sum_{i=2}^{r} |V(C_i)|$$

$$\leq n + 1 + t + n - 1 - t = 2n,$$

contradicting the hypothesis that G is n-extendable.

Thus we may assume that all components of $G - S$ are singletons, say $z_1, \ldots, z_\alpha$, where $\alpha \geq 2$. But then we have $2n < |V(G)| = n+1+\alpha$, or $\alpha > n - 1$, that is, $\alpha \geq n$.

Now by Theorem 1980A, graph G is 1-extendable and so the single line ab extends to a perfect matching F_2 of G which must match the set $\{z_1, \ldots, z_\alpha\}$ into $\{u_1, \ldots, u_{n-1}\}$ and hence $\alpha \leq n - 1$, a contradiction.

This completes the proof of part (a).

To prove part (b), suppose once again that $C_1, \ldots, C_r$ are the components of $G - S$, that $|V(C_1)| \geq |V(C_j)|$ for $j > 1$ and that $r \geq n + 1$.

Suppose that $|V(C_1)| \geq 2$. Since S is a minimum cutset in G, by the preceding Lemma we may match u_1 and u_2 into $V(C_1)$, and u_j (if any) into C_{j-1} for $3 \leq j \leq n$. Extend this matching to a perfect matching F_3 of G.

First suppose that F_3 matches u_{n+1} into C_1. Then $C_n, \ldots, C_r$ must all be even and $C_1, \ldots, C_{n-1}$ all odd. On the other hand, again because S is a minimum cutset, we may match u_1 into C_2, u_2 into $C_3, \ldots$, and u_n into C_{n+1}. Extend this matching to a perfect matching F_4 of G. Then F_4 must match u_{n+1} into C_1, since C_1 is odd. But then C_n must be odd, a contradiction.

So we may assume that F_3 does not match u_{n+1} into C_1. We have two cases to consider.

Case 1. Suppose $n = 2$. Without loss of generality, assume that F_3 matches $u_{n+1} = u_3$ into C_2. Then $C_1, C_3, \ldots, C_r$ are all even, but C_2 is odd. Now form a different matching which matches u_1 to C_1 and u_2 to C_3 and extend to a perfect matching F_5 of G. Then F_5 must match point u_3 to even component C_1. But then it follows that C_2 must be even, a contradiction.

Case 2. Suppose $n \geq 3$. There are two subcases to consider.

First, suppose F_3 matches u_{n+1} into some C_j, where $2 \leq j \leq n-1$. Without loss of generality, assume that u_{n+1} is matched into C_2. Then $C_1, C_2, C_n, \ldots, C_r$ are all even, while the rest of the C_j's, if any, are all odd. Now construct a new matching taking u_1 to C_n and u_2 to C_{n+1} and leaving $u_3, \ldots, u_n$ as matched by F_3 above. Extend to a perfect matching F_6 of G. But then it follows that since C_2 is even, F_6 must match u_{n+1} into C_2. But then component C_n must be odd and again we have a contradiction.

Second, suppose that F_3 matches u_{n+1} into some C_j with $j \geq n$. Renumbering if necessary, we may suppose that $j = n$. Thus components $C_1, C_{n+1}, \ldots, C_r$ must all be even, while $C_2, \ldots, C_n$ are all odd.

Now construct yet another matching pairing u_1 to C_{n+1} and leaving $u_2, \ldots, u_n$ as matched by F_3. Extend this to a perfect matching F_7 of G. But then since C_1 is even, F_7 must match u_{n+1} to C_1. But then C_n must be even, a contradiction.

Thus we may assume that $|V(C_1)| = \cdots = |V(C_r)| = 1$. But then if $r > n + 1$, graph G cannot have a perfect matching, so we have that $r = n + 1$.

FIGURE 2.

But again since G is $(n + 1)$-connected, we have $\deg v \geq n + 1$ for all $v \in V(G)$ and hence $G = K_{n+1,n+1}$.

Trivially, if $G = K_{n+1,n+1}$ and S is any minimum cutset of points in G, then S must be one of the two classes of the bipartition and hence $G - S$ has precisely $n + 1$ (singleton) components.

This completes the proof of the theorem.   ∎

REMARK 1.   The restriction in part (b) of the theorem above that $n \geq 2$ is necessary in the following sense. If $n = 1$, there are infinitely many 1-extendable graphs with cutsets S such that $|S| = 2$, but having 2 *or more* components in $G - S$. Moreover, such components may be non-trivial. In Figure 2, we show an infinite family of 1-extendable graphs each having a cutset S of size 2, but having the number k of components of $G - S$ as large as one likes. (Note: The large plus sign in Figure 2 and in subsequent figures in this paper signifies the "join" operation where all points on the left are joined to all points on the right.)

REMARK 2.   The examples in Figure 2 show that 1-extendable graphs may have arbitrarily small *toughness*. (Recall that the **toughness** of a graph G is defined by $\min(|S|/(c(G - S))$ where S ranges over all cutsets of G and $c(G - S)$ denotes the number of components of $G - S$.)

FIGURE 3. The family $\{G_n\}$

For general n, the toughness of an n-extendable graph may be arbitrarily small as well. In Figure 3 we show an infinite family of graphs $\{G_n\}$, $n = 1, 2, \ldots$, where for each n, graph G_n is n-extendable, but the toughness of $G_n = t(G_n) \leq 2n/(2n+k)$, where k can be any positive integer chosen as large as one likes.

REMARK 3. At the other extreme, there are n-extendable graphs with minimum cutsets S of size $n + 1$, but where $G - S$ has only two components. Figure 4 shows an infinite family of such graphs $\{H_n\}$, $n \geq 1$, where $\kappa(H_n) = n + 1$. This family also shows that the *line-*connectivity of an n-extendable graph can be arbitrarily large, while the point-connectivity remains at its minimum value $n + 1$. In this family, k may denote any odd positive integer. It is a tedious, though straightforward, argument to show that each H_n is n-extendable.

REMARK 4. There are n-extendable graphs which not only have point-connectivity $n + 1$, but even have minimum degree $n + 1$. (Hence $\kappa(G) = \lambda(G) = \text{mindeg}(G) = n + 1$.) For each $n \geq 1$, a family of such graphs (which are, in fact, bipartite) is given by $B_n = K_{n+2,n+2} - F$, where F is a perfect matching.

Let us now recall the definition of *local connectivity* of a graph. A

FIGURE 4.   The family $\{H_n\}$

graph G is said to be **locally connected** if for every point $v \in G$, the induced subgraph $G[\Gamma(v)]$, i.e., the subgraph induced by the (deleted) neighborhood of v, is connected. It was first shown by Chartrand and Pippert (1974) that neither the property of being connected nor the property of being locally connected implies the other. (See also Vanderjagt (1974).)

The following result involves connectivity, local connectivity and n-extendability, and is an immediate consequence of Theorem 2.2.

2.3. THEOREM.   *If G is n-extendable $(n \geq 1)$ and locally connected, then G is $(n+2)$-connected. Moreover, this lower bound on the connectivity of G is sharp for all n.*

PROOF.   We know that G is $(n+1)$-connected by Theorem 1930B, so suppose that $\kappa(G) = n + 1$ and that $S = \{u_1, \ldots, u_{n+1}\}$ is a minimum cutset. By Theorem 2.2, this cutset S is independent. But then for every $u \in S$, $G[\Gamma(u)]$ is disconnected, a contradiction.

A family of extremal graphs $\{M_n\}_{n=1}^{\infty}$ is shown in Figure 5.

It is easy to see that for each n, graph M_n is n-extendable, locally connected and has $\kappa(M_n) = n + 2$. We hasten to point out that the extremal graphs in this family are not the *smallest* to be had, but it is

FIGURE 5. The family $\{M_n\}$

especially easy to prove that these particular graphs are n-extendable. ∎

3. Removing and Adding Lines to an n-extendable Graph

The next theorem treats the effect on matching extendability of removing a line.

3.1. THEOREM. *Suppose G is n-extendable, for some $n \geq 1$. Then if e is any line in G:*
  (a) *if $n = 1$, $G - e$ has a perfect matching, while*
  (b) *if $n \geq 2$, $G - e$ is $(n-1)$-extendable.*

PROOF. If G is 1-extendable, then G is connected and $|V(G)| \geq 4$. Thus there must be a line f adjacent to line e. Extend f to a perfect matching F_1 for G and note that F_1 cannot contain e. Thus $G - e$ has a perfect matching and (a) is proved.

Now suppose that $n \geq 2$. Then by Theorem 1980A, graph G is 1-extendable and by part (a) graph $G - e$ has a perfect matching.

Now suppose $X = \{e_1, \ldots, e_{n-1}\}$, where $e_i = a_i b_i$, is a set of $n - 1$ independent lines in $G - e$ which does not extend to a perfect matching of $G - e$. Again by Theorem 1980A, we know that G is $(n-1)$-extendable and so X extends to a perfect matching F_2 of G. Hence we may suppose that every perfect matching for G which contains X also contains line e. In particular, $X + e$ is an independent set of lines in G.

Now Let $A = \{a_1, \ldots, a_{n-1}\}$, $B = \{b_1, \ldots, b_{n-1}\}$, and let $e = ab$. Suppose that $\Gamma_{G-e}(a) \cup \Gamma_{G-e}(b) \not\subseteq A \cup B$; say for example, that a is adjacent to point c where $c \notin A \cup B$. Then $X \cup \{ac\}$ is a set of n independent lines in $G - e$ containing X and thus by the hypothesis of this theorem, $X \cup \{ac\}$ extends to a perfect matching F_3 of G which cannot contain e. But this contradicts the definition of X.

So we may suppose that $\Gamma_{G-e}(a) \cup \Gamma_{G-e}(b) \subseteq A \cup B$. From now on, we may assume $n \geq 3$, for if $n = 2$, either $\{a_1, b_1\}$ is a cutset for G, contradicting the fact that (by Theorem 1980B) $\kappa(G) \geq 3$, or else $|V(G)| = 4 < 6 = 2n + 2 \leq |V(G)|$ which is also a contradiction.

Now let $S = A \cup B$. Since $|S \cup V(e)| = |S| + 2 = 2n$, and $|V(G)| \geq 2n + 2$, we must have that $T = G - (S \cup V(e)) \neq \emptyset$.

Now S is a cutset of G, separating a and b from T. (See Figure 6.) But G is $(n + 1)$-connected by Theorem 1980B so at least $n + 1$ points in S are adjacent to points in T.

Let us call points a_i and b_i **mates** of each other for $i = 1, \ldots, n-1$. Suppose a point α of S is adjacent to a point of T via a line g. We claim that β, the mate of α, does not lie in $\Gamma_{G-e}(a) \cup \Gamma_{G-e}(b)$. Suppose, to the contrary, that $\beta \in \Gamma_{G-e}(a)$, say. Then if $\alpha \in V(e_i)$, say, we have a set $X - e_i + g + a\beta$ of n lines which are independent in $G - e$, and hence in G. Thus this set extends to a perfect matching F_4 of G. But F_4 cannot cover point b and we have a contradiction.

Thus whenever a point α of S is adjacent to a point of T, its mate $\beta \notin \Gamma_{G-e}(a) \cup \Gamma_{G-e}(b)$. But since $\kappa(G) \geq n + 1$, at least $n + 1$ points of S are adjacent to points of T and so at least $n + 1$ points of S are not in $\Gamma_{G-e}(a) \cup \Gamma_{G-e}(b)$. Hence at most $2n - 2 - (n + 1) = n - 3$ points of S are in $\Gamma_{G-e}(a) \cup \Gamma_{G-e}(b)$. Hence $\kappa(G) \leq n - 3$, a contradiction of Theorem 1980B.   ∎

If one removes a matching of size 2 or more from an n-extendable graph, one cannot hope to retain $(n-1)$-extendability in general. Consider the graph B in Figure 7.

This bipartite graph B is 2-extendable, but $B - u_1 v_2 - u_2 v_1$ is not 1-

FIGURE **6.**

FIGURE **7.** The graph B

extendable, for line $u_4 v_4$ will not extend to a perfect matching.

However, if one removes a matching of size 2 the lines of which are not "far apart", we can preserve $(n-1)$-extendability. More precisely, we have the following result.

3.2. THEOREM. *Suppose $n \geq 2$, graph G is n-extendable and $\{e, f\}$ is a set of two independent lines joined by at least one other line g. (That is, egf is a path of length 3.) Then $G - e - f$ is $(n - 1)$-extendable.*

PROOF. Note first that $G - e - f$ contains a perfect matching because G is 1-extendable and hence contains a perfect matching which includes line g.

Now suppose X is a set of $(n - 1)$ independent lines in $G - e - f$. We know that X extends to a perfect matching F_1 for G since G is $(n - 1)$-extendable. So if $F_1 \cap \{e, f\} = \emptyset$, we are done.

Hence suppose that for every perfect matching F_1 of G which contains X, we have $F_1 \cap \{e, f\} \neq \emptyset$. Let F_1 be such a perfect matching and suppose, without loss of generality, that $e \in F_1$.

First suppose also that f is adjacent with a line of X, say x_1. By Theorem 3.1, there is a perfect matching F_2 for $G - e$ which contains X. But then F_2 cannot contain f and hence F_2 is a perfect matching for $G - e - f$ containing X.

So now suppose that $X \cup \{e, f\}$ is a set of $n + 1$ independent lines. But then $X \cup \{g\}$ is a set of n independent lines which must, therefore, extend to a perfect matching F_3 of G. But $F_3 \cap \{e, f\} = \emptyset$ and the proof is complete. ∎

It is interesting to contrast the effect of *deleting* a line from an n-extendable graph with the effect of *adding* a line not previously present. Actually, if G is n-extendable and one adds a new line e, it may happen that the new graph $G + e$ is not even 1-extendable! For example, for every $n \geq 1$ the complete bipartite graph $K_{n+1,n+1}$ is n-extendable, but if one adds a new line e joining two points in the same set of the bipartition, that line e clearly cannot lie in any perfect matching of the graph $K_{n+1,n+1} + e$.

References

J. A. BONDY AND U. S. R. MURTY

1976.   *Graph Theory with Applications*, American Elsevier, New York, 1976.

R. A. BRUALDI AND H. PERFECT

1971.   Extension of partial diagonals of matrices I, *Monatsh. Math.* **75**, 1971, 385–397.

G. CHARTRAND AND R. E. PIPPERT

1974.   Locally connected graphs, *Časopis Pěst. Mat.* **99**, 1974, 158–163.

J. EDMONDS

1965.   Paths, trees and flowers, *Canad. J. Math.* **17**, 1965, 449–467.

J. EDMONDS, L. LOVÁSZ AND W. R. PULLEYBLANK

1982.   Brick decompositions and the matching rank of graphs, *Combinatorica* **2**, 1982, 247–274.

T. GALLAI

1963.   Kritische Graphen II, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8**, 1963, 373–395.

1964.   Maximale Systeme unabhängiger Kanten, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9**, 1964, 401–413.

D. J. HARTFIEL

1970.   A simplified form for nearly reducible and nearly decomposable matrices, *Proc. Amer. Math. Soc.* **24**, 1970, 388–393.

G. HETYEI

1964.   Rectangular configurations which can be covered by 2×1 rectangles, *Pécsi Tan. Főisk. Közl.* **8**, 1964, 351–367. (Hungarian).

A. KOTZIG

1959a.   On the theory of finite graphs with a linear factor I, *Mat.-Fyz. Časopis Slovensk. Akad. Vied* **9**, 1959, 73–91. (Slovak).

1959b.   On the theory of finite graphs with a linear factor II, *Mat.-Fyz. Časopis Slovensk. Akad. Vied* **9**, 1959, 136–159. (Slovak).

1960.   On the theory of finite graphs with a linear factor III, *Mat.-Fyz. Časopis Slovensk. Akad. Vied* **10**, 1960, 205–215. (Slovak).

L. LOVÁSZ

1972.   On the structure of factorizable graphs, *Acta Math. Acad. Sci. Hungar.* **23**, 1972, 179–195.
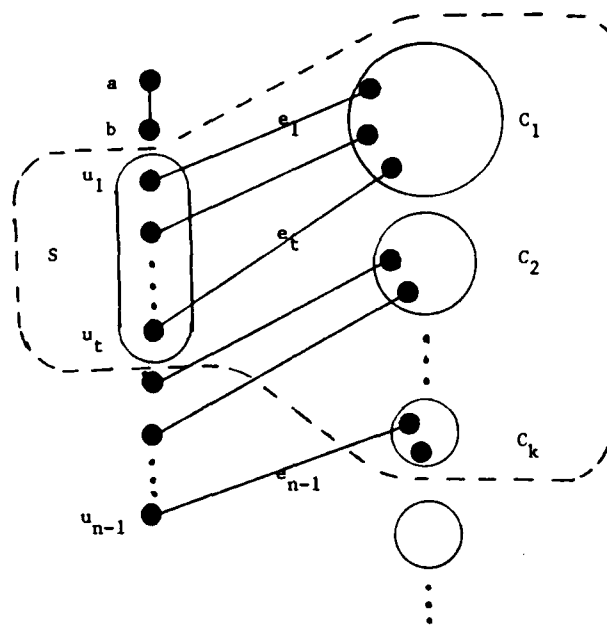
L. Lovász and M. D. Plummer

1977.   On minimal elementary bipartite graphs, *J. Combin. Theory Ser. B* **23**, 1977, 127–138.

1986.   *Matching Theory*, Ann. Discrete Math., North-Holland, Amsterdam, 1986 (to appear).

M. D. Plummer

1980.   On n-extendable graphs, *Discrete Math.* **31**, 1980, 201–210.

1985.   A theorem on matchings in the plane, *Conference in memory of Gabriel Dirac*, Ann. Discrete Math., North-Holland, Amsterdam, to appear.

1986a.   Matching extension in bipartite graphs, preprint, 1986.

1986b.   Toughness and matching extension in graphs, preprint, 1986.

1986c.   Matching extension and the genus of a graph, 1986, submitted for publication.

D. W. Vanderjagt

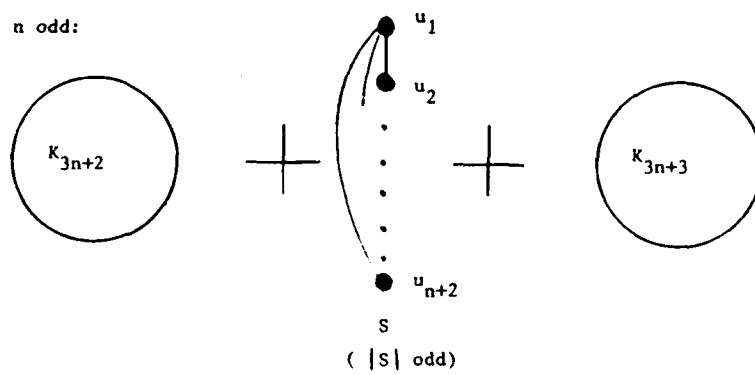1974.   Graphs with prescribed local connectivities, *Discrete Math.* **10**, 1974, 391–395.

S $\quad\bullet\quad\bullet\quad\bullet\quad\}$ k independent lines
$(k \geq 2)$

2

$$S$$
$$|S| = 2n$$

$$+$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} \begin{array}{l} \text{2n+k copies of } K_2 \\ (k > 0) \end{array}$$

$$K_{4nk} \quad + \quad \vdots \quad + \quad K_{3nk+1}$$

S

$|S| = n+1$

n odd:

K_{3n+2} $+$ u_1 u_2 $\vdots$ u_{n+2} $+$ K_{3n+3}

S

(|S| odd)

n even:

K_{3n+2} $+$ u_1 u_2 $\vdots$ u_{n+2} $+$ K_{3n+2}

S

(|S| even)

5

$$S = A \cup B$$

B:

u_1 v_1
u_2 v_2
u_3 v_3
u_4 v_4

END

DTIC

7 - 86